where Σ' denotes summation over only those terms for which $(n + r)$ is even. A one term approximation for Γ yields

$$A_1 = \pi\alpha/(\pi/2 + 16s/3\pi C) \tag{14}$$

A two term approximation shows

$$\begin{aligned} A_1 &= \pi\alpha\Big/\left[\left(\pi/2 + \frac{16s}{3\pi C}\right) - \left(\frac{16s}{15\pi C}\right)^2\Big/\left(\frac{\pi}{2} - \frac{144s}{35\pi C}\right)\right] \\ A_2 &= 0 \\ A_3 &= \frac{16s\alpha}{15C}\Big/\left[\left(\frac{\pi}{2} + \frac{16s}{3\pi C}\right)\left(\frac{\pi}{2} - \frac{144s}{35\pi C}\right) - \left(\frac{16s}{15\pi C}\right)^2\right] \end{aligned} \tag{15}$$

The computation effort required in solving Eq. (13) is less compared to a collocation method where trigonometric functions must be evaluated.

Conclusions

The method outlined in this note is valid for any set of loadings (Γ_n, α_n) which are orthogonal in Graham's sense. It may be used for non-orthogonal loadings if they are first converted to an orthogonal set as suggested by Graham.[3]

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Generalization of Dunkerley's Formula for Finding the Lowest Critical Velocity of Rotating Shafts

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THE well-known Dunkerley's method is widely used for finding an approximate value of the first critical velocity of forward precession of rotating shafts.[1-3] It is based on the mathematical property of algebraic equations such that if

$$\alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0 \tag{1}$$

is an algebraic equation, then

$$\sum_{i=1}^{n}\lambda_i = -\frac{\alpha_1}{\alpha_0} \tag{2}$$

In the case of critical velocities of a rotating shaft with n masses, when the damping effects and the weight of the shaft are neglected, the characteristic Eq. (1) is given by

$$\begin{vmatrix} a_{11}m_1 - \lambda & a_{12}m_2 & \cdots & a_{1n}m_n \\ a_{21}m_1 & a_{22}m_2 - \lambda & \cdots & a_{2n}m_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}m_1 & a_{n2}m_2 & \cdots & a_{nn}m_n - \lambda \end{vmatrix} = 0 \tag{3}$$

where m_i are the concentrated masses, a_{ik} the flexibility factors and the critical velocities ω_i are given by $\omega = (1/\lambda)^{1/2}$. Equation (2) now is written

$$\sum_{i=1}^{n}\frac{1}{\omega_i^2} = \sum_{i=1}^{n}\lambda_i = -\frac{\alpha_1}{\alpha_0} = \sum_{i=1}^{n}a_{ii}m_i = \sum_{i=1}^{n}\frac{1}{\bar{\omega}_i^2}, \tag{4}$$

where $\bar{\omega}_i^2$ is the square of the critical velocity of the shaft with the mass m_i alone.

Because $\omega_1 \ll \omega_2 \ll \cdots \omega_n$, it's possible to write[1,2]

$$\frac{1}{\omega_1^2} \cong \sum_{i=1}^{n}\frac{1}{\omega_i^2} = \sum_{i=1}^{n}\frac{1}{\bar{\omega}_i^2} \tag{5}$$

This is Dunkerley's formula in the usual form. It gives a very good approximation of the first critical velocity when the lateral inertia of the masses is not important.[2,3]

When the lateral inertia of m of the masses is considerable, the characteristic Eq. (1) is given by

$$\begin{vmatrix} a_{11}m_1 - \lambda & a_{12}m_2 & \cdots & a_{1n}m_n - b_{11}D_1 - b_{12}D_2 & \cdots & -b_{1m}D_m \\ a_{21}m_1 & a_{22}m_2 - \lambda & \cdots & a_{2n}m_n - b_{21}D_1 - b_{22}D_2 & \cdots & -b_{2m}D_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}m_1 & a_{n2}m_2 & \cdots & (a_{nn}m_n - \lambda) - b_{n1}D_1 - b_{n2}D_2 & \cdots & -b_{nm}D_m \\ a_{11}'m_1 & a_{12}'m_2 & \cdots & a_{1n}'m_n(-b_{11}'D_1 - \lambda) - b_{12}'D_2 & \cdots & -b_{1m}'D_m \\ a_{21}'m_1 & a_{22}'m_2 & \cdots & a_{2n}'m_n - b_{21}'D_1(-b_{22}'D_2 - \lambda) & \cdots & -b_{2m}'D_m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}'m_1 & a_{m2}'m_2 & \cdots & a_{mn}'m_n - b_{m1}'D_1 - b_{m2}'D_2 & \cdots & -b_{mm}'D_m - \lambda \end{vmatrix} = 0 \tag{6}$$

where a_{ik}, b_{ik}, a'_{ik}, b'_{ik} are flexibility factors and D_k are the differences between the moment of inertia of each mass with respect to the axis tangent to the elastic line and the moment of inertia with respect to the axis perpendicular to the elastic line.[2] If all the D_k are positive (that is, if there are m discs and $n - m$ concentrated masses), Eq. (6) has n real positive roots and m real negative roots.[2,5]

In this case Eq. (2) is written, taking into account the lateral inertia

$$\sum_{i=1}^{n+m}\frac{1}{\omega_i^2} = \sum_{i=1}^{n}a_{ii}m_i + \sum_{i=1}^{m}(-b_{ii}'D_i) = \sum_{i=1}^{n+m}\frac{1}{\bar{\omega}_i^2} \tag{7}$$

where $\bar{\omega}_i^2$ is now the square of the critical velocity of the shaft with the mass m_i alone (positive) or with the lateral inertia D_i alone (negative).

If $|\omega_1^2| \ll |\omega_2^2|$ it's possible to write

$$\frac{1}{\omega_1^2} \cong \sum_{i=1}^{n+m}\frac{1}{\omega_i^2} = \sum_{i=1}^{n+m}\frac{1}{\bar{\omega}_i^2} \tag{8}$$

This formula gives a very good approximation of the first critical velocity of a rotating shaft except when $|\omega_1^2| \cong |\omega_2^2|$, that is, when the first real and the first imaginary critical speeds are similar in modulus, a situation which also causes the classical iterative methods to fail.[4]

Example

As an example, a real shaft with two supports was analyzed. For the computation, the real shaft was subdivided into ten sections, and for two of them (representing compressor and turbine), the lateral inertia was taken into account. The supports were at the left end of the shaft and between the 6th and 7th section.

The compressor was substituted by the 3rd section and the turbine by the 10th section. The span between the two supports was 24.5 cm and the overspan between the sec-

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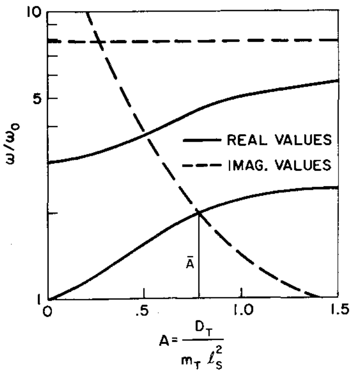

Fig. 1 First four critical speeds.

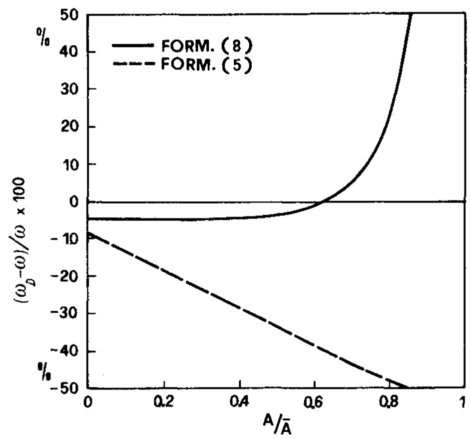

Fig. 2 Approximation of Dunkerley's formula.

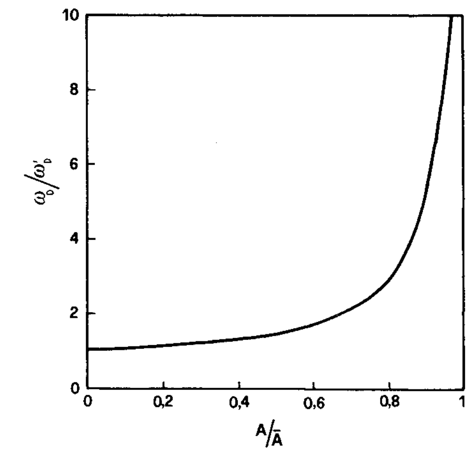

Fig. 3 Ratio between ω_D formula (8) and ω'_D formula (5).

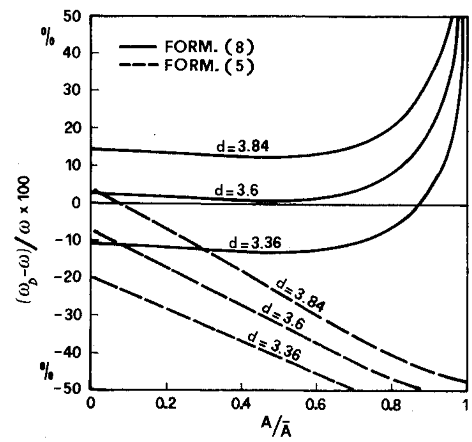

Fig. 4 Approximation of Dunkerley's formula (simplified system).

ond support and the turbine 6.1 cm. The diameter of the shaft was variable between 2.8 and 4.4 cm. The lateral inertia of the compressor was kept constant ($D_c = 0.25$ kg cm sec^2), while that of the turbine was taken as variable.

Figure 1 shows the variation of the first four critical velocities as functions of the lateral inertia D_T of the turbine. Solid lines refer to real ω and dashed lines to imaginary ω. The critical velocities were nondimensionalized using the first critical velocity corresponding to the case without lateral inertia of the turbine. The horizontal axis represents the ratio A between the lateral inertia D_T and the mass m_T of the turbine multiplied by the square of the length l_s of the overspan.

In Fig. 2, the approximations obtained using formula (5) and formula (8) are compared. The vertical axis represents the ratio between the error ($\omega_D - \omega$) and the exact value ω of the critical velocity; ω_D is the critical velocity obtained using either formula (5) or (8). The horizontal axis represents the ratio between $A = D_T/m_T l_s^2$ and the value $\bar{A}$ for which the first real and the first imaginary critical velocity are coincident in modulus (see Fig. 1).

Figure 3 shows the ratio between ω_D found using formula (8), and ω'_D found using formula (5). The slope of this curve may be taken as an index for the applicability of formula (8). For values of A larger than $\bar{A}$, formula (8) will give an approximation of the first imaginary ω, while formula (5) will give always an approximation of the first real ω.

The improvement of the approximation using formula (8) instead of formula (5) also is considerable if those formulas are applied to a very simplified system, taking into account just the masses and the lateral inertias of the compressor and of the turbine, and assuming a constant diameter for computing the elasticity of the shaft. Figure 4 shows the approximation obtained in this case using three different values of the diameter of the shaft. The values ω_D computed using formulas (5) and (8) are again referred to the values of ω_0 represented in Fig. 1.

Conclusions

The application of Dunkerley's formula to the determination of the first critical speed of rotating shafts when the lateral inertia is not negligible has been analyzed. It has been shown that the use of formula (5) gives, in this case, just a poor approximation of the exact value, while a strong improvement can be obtained using the more general formula (8). It is concluded that, using the generalized formula (8) it's possible to obtain a quite satisfactory value of the first critical velocity also when the lateral inertia of some of the masses is important.

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